On Attractors of the Viscous Cahn-Hilliard Equation

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1 Background

The viscous Cahn-Hilliard equation arises in the dynamics of viscous phase transitions in cooling binary solutions. In the last 20 years there has been an extensive amount of literature regarding the viscous Cahn-Hilliard equation published. In [1, 6], the authors show the global existence on bounded domains. As a result the dynamics are well established and understood. More recently, [11] investigates global dynamic for the semiflow bounded domains. The authors are able to show show that the 3D viscous Cahn-Hilliard equation has a global attractor in H^4 when the initial value belongs to H^1. In [3, 4], the authors investigate the viscous Cahn-Hilliard equation on unbounded domains. Under certain conditions, global existence and existence of global attractors has been proven. For this assignment, the work in [4] is outlined and then bounds on this attractor are found under suitable conditions.

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. The viscous Cahn-Hilliard equation, with Dirichlet boundary conditions is,

$$\begin{aligned} &(1-\nu)u_t = -\Delta(\Delta u + f(x,u) - \nu u_t), \quad t>0,\ x\in\Omega \\ &u(0,x) = \Delta u(0,x) = 0, \quad x \in \partial\Omega \\ &u(0,x) = u_0(x) \end{aligned} \tag{1.1}$$

For a parameter $\nu \in [0,1]$. For $\nu = 1$ this a semilinear heat equation, for $\nu = 0$ this is the Cahn-Hilliard equation. Considering $n \geq 3$, the following assumptions will be made about the non-linearity $f(x,u)$. It will be of the form $f(x,u) = g(x,u) - \alpha u$ where $\alpha > 0$. $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous, $g(x,0) \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^2(\mathbb{R}^n)$ and $g(x,u)$ holds a locally Lipschitz condition,

$$|g(x,u_1) - g(x,u_2)| \leq L|u_2 - u_2|(\psi(x) + |u_1|^q + |u_2|^q) \tag{1.2}$$

for a non-negative function $\psi(x) \in \mathcal{L}^n(\mathbb{R}^n)$, and $q \in [0, 4/(n-2)]$. Also, if $G(x,u)$ is a primitive for $g(x,u)$ in the variable u the following estimate holds:

$$G(x,s) \leq \mu s^2 + C_\mu(x)|s| + \phi(x), \tag{1.3}$$

where $\mu \in (0, \alpha/2)$, $C_\mu(x) \in \mathcal{L}^2(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ and $\phi(x) \in \mathcal{L}^1(\mathbb{R}^n)$ and non-negative. From the literature, a typical non-linearity is of the form

$$f(x,u) = -\alpha u + \gamma u^3, \quad \alpha, \gamma > 0. \tag{1.4}$$

As the unbounded case poses slightly more problems, I will focus on the Cauchy problem.

$$\begin{aligned} &(1-\nu)u_t = -\Delta(\Delta u + f(x,u) - \nu u_t), \quad t > 0,\ x \in \mathbb{R}^n \\ &u(0,x) = u_0(x) \end{aligned} \tag{1.5}$$

One of the first investigations into global existence was done in [2]. The authors show a prior estimates in $\mathcal{L}^\infty(\mathbb{R}^2)$ under reasonable conditions. In

2 Existence of Solutions

The following is a summary from [4], For a fixed $\nu \in (0,1)$, let $A_\nu = ((1-\nu)I - \nu\Delta)$. Then the domain of $\text{dom}(A_\nu) = H^2(\mathbb{R}^n)$. Also A_ν is a self-adjoint operator in $L^2(\mathbb{R}^n)$ with$\sigma(A_\nu) \subset [1-\nu, \infty)$. Hence A_ν^{-1} is a linear operator on $\mathcal{L}^2(\mathbb{R}^n)$.

2.1 Local Existence

Let

$$B_\nu = \frac{A_\nu}{\nu^2} - \frac{2-2\nu}{\nu^2} I + \frac{(1-\nu)^2}{\nu^2} A_\nu^{-1} \tag{2.1}$$

where I is identity operator. The equation 1.5 can be written as

$$u_t = -B_\nu u + \left(\frac{1}{\nu} - \frac{I-\nu}{v}\right) f(x,u) = -B_\nu + F(x,u) \tag{2.2}$$

Where the non-linearity $F = (I - (1-\nu))f(\cdot,u)/\nu$ acts from $X_1 = H^1(\mathbb{R}^n)$ into $Y_1 = H^{\epsilon-1}(\mathbb{R}^n)$ is Lipschitz continuous on bounded sets with $0 < \epsilon$ small. For notation, let $Y_k = H^{\epsilon-k}(\mathbb{R}^n)$. Let $\nu \in (0,1)$, $n \geq 3$ and let $B \subset H^1(\mathbb{R}^n)$ be bounded. If u_1, $u_2 \in B$, then we have the estimate, see (7) in [4]

$$\|F(u_1) - F(u_2)\|_Y \leq C(B)\|u_1 - u_2\|_{X_1} \tag{2.3}$$

For $\alpha = 1 - \epsilon/2$, we have a local solution given by

$$u(t) = e^{-B_\nu t}u_0 + \int_0^{t_0} e^{-B_\nu(t-s)} F(u(s))\, ds. \tag{2.4}$$

Where $e^{-B_\nu t}$ is semigroup corresponding to the operator B_ν in Y. A similar result can be shown for $\nu = 0$. As we have the estimate

$$\|F(u_1) - F(u_2)\|_{Y_3} \leq c\|f(\cdot,u_1) - f(\cdot,u_2)\|_{Y_1} \tag{2.5}$$

Using a bootstrapping argument, if the non-linearity f is smooth enough it is shown in [4] that the solution has arbitrary finite H^k regularity for small t. We have the estimate, (see (10) in [4])

$$\|u\|_{H^{1+\epsilon^-}} \leq C_\nu t^{-\epsilon^-/2}\|u_0\|_{H^1} + \int_0^{t_0} \frac{c_\nu}{(t-s)^{1-\delta/2}} \|F(u(s))\|_{Y_1}, \tag{2.6}$$

if $f \in C^2, 3 \leq n \leq 6$, with the following assumptions on f; For all $\eta > 0$

$$\begin{aligned} &\exists C_\eta(x) \in \mathcal{L}^n(\mathbb{R}^n), \quad |\partial_{x_i}\partial_u g(x,u)| \leq \eta|u|^{6/(n-2)} + C_\eta(x) \\ &\exists C_\eta(x) \in \mathcal{L}^\infty(\mathbb{R}^n), \quad |\partial_u^2 g(x,u)| \leq \eta|u|^{(6-n)/(n-2)} + C_\eta(x) \\ &\exists C_\eta(x) \in \mathcal{L}^\infty(\mathbb{R}^n), \quad |\partial_{x_i}^2 g(x,u)| \leq \eta|u|^{(6+n)/(n-2)} + C_\eta(x) \end{aligned} \tag{2.7}$$

Lemma 1 *When f satisfies 2.7 then the local $H^1(\mathbb{R}^n)$ solution is bounded, for $t > 0$ and as long as it exists, in $H^2(\mathbb{R}^n)$ and $H^3(\mathbb{R}^n)$.*

The proof uses standard tricks, multiply by 1.5 by Δu, using the assumptions of f to bound its derivatives combination of Agmons inequality, Sobolev embedding and interpolation, see Lemma 1 in [4] for exact details. The final estimates are given below. Define $T_1(u)$ as follows,

$$T_1(u) = (1-\nu)\int_{\mathbb{R}^n} |\nabla u(t,\cdot)|^2\,dx + \nu\int_{\mathbb{R}^n} |\Delta u(t,\cdot)|^2\,dx \tag{2.8}$$

First,

$$\frac{d}{dt}T_1(u) \leq -cT_1(u) + C_1(\|u_0\|_{H^1}), \tag{2.9}$$

where c is constant that depends on n, α and $C_1(\|u_0\|_{H^1})$ is independent of ν. Second, define $T_2(u)$

$$T_2(u) = (1-\nu)\int_{\mathbb{R}^n} |\Delta u(t,\cdot)|^2\,dx + \nu\int_{\mathbb{R}^n} |\nabla\Delta u(t,\cdot)|^2\,dx \tag{2.10}$$

Then

$$\frac{d}{dt}T_2(u) \leq -cT_2(u) + C_1(\|u_0\|_{H^1}), \tag{2.11}$$

where again c is constant that depends on n, α and $C_2(\|u_0\|_{H^1})$ is independent of ν.

2.2 Global Existence

Continuing to follow the work presented in [4], let $\omega = \Delta u + f(x,u) - \nu u_t$. Multiplying 1.5 by ω and integrating over $\mathbb{R}^n$, we have that

$$(1-\nu)\int_{\mathbb{R}^n} \omega u_t\,dx = \int^{\mathbb{R}^n} |\nabla\omega|^2\,dx \geq 0 \tag{2.12}$$

By local existence, $|\nabla\omega|$ is defined for $t > 0$ as long as the H^1 solution exists. Using 2.11 the conditions that the local solutions satisfies to get 2.4, we have

$$\frac{d}{dt}\left(\|\nabla u\|^2_{\mathcal{L}^2} + \alpha\|u\|^2_{\mathcal{L}^2} - 2\int_{\mathbb{R}^n} G(x,u)\,dx\right) + \nu(1-\nu)\|u_t\|^2 \leq 0 \tag{2.13}$$

By the assumptions on g, we have the estimate

$$|G(x,u)| \leq \frac{L}{2}\psi(x)|u|^2 + \frac{L}{q+2}|u|^{q+2} + |g(x,0)||u| \tag{2.14}$$

Using the previous and 2.13 we can obtain an a priori bound that only depends

$$\|\nabla u(t)\|^2_{\mathcal{L}^2} + c(\alpha,\mu)\|u(t)\|^2_{\mathcal{L}^2} \leq P(u_0, g(x,0), \phi(x), \psi(x)) < \infty \tag{2.15}$$

where $c(\alpha,\mu) > 0$ is a constant and P is a quadratic polynomial that depends on the various norms of it's arguments and constants α, μ, L, q, n, but is independent of t, see (30) in [4] for exact bound. As a consequence of this estimate local solutions can be extended globally in time. The work required for this requires estimates on F; namely, for $\nu \in (0,1)$

$$\|F(u(t))\|_{Y^1} \leq K(\|u(t)\|_{H^1}, \tag{2.16}$$

Where $K : [0,\infty) \to [\infty)$ is non-decreasing. For $\nu = 0$,

$$\|F(u(t))\|_{Y^3} \leq K''(\|u(t)\|_{H^1}, \|f(\cdot,0)\|_{\updownarrow^2}) \tag{2.17}$$

See (31), (32) in [4]. For $\nu \in [0,1)$, 1.5 defines a semigroup $S(t)$ on the phase space of $H^1(\mathbb{R}^n)$. So for $t \geq 0$, we have $S(t)u_0 = u(t)$ where $u(t)$ is the global in time H^1 solution.

3 Global Attractor

A global attractor is found by a Lyapunov and estimating the stationary solutions. Once established, we have 1.5 gives rise to a gradient system and the existence of the global attractor follows, see [9].

3.1 Lyapunov function

Let $L : H^1(\mathbb{R}^n) \to \mathbb{R}$, be defined by,

$$L(u) = \alpha\|u\|^2_{\mathcal{L}^2} + \|\nabla u\|^2_{\mathcal{L}^2} - 2\int_{\mathbb{R}^n} G(x,u)\,dx, \tag{3.1}$$

the goal is to show that this is a Lyapunov function for 1.5. Observe the following:

a. With the the assumptions 1.2, 1.3 we have that L is bounded from below and L is continuous on H^1.

b. From 2.13, it follows that $\frac{d}{dt}L(u(t)) \le 0$; hence, For each $u \in H^1$, there is a the function $t \mapsto L(S(t)u) \in \mathbb{R}$ is non-increasing.

c. A simple computation reveals that for $t > 0$,

$$\frac{d}{dt}L(u(t)) = -\frac{2}{1-\nu}\int_{\mathbb{R}^n} |\nabla\omega|^2\,dx - 2\nu\|u_t(t)\|^2_{\mathcal{L}^2} \le 0. \tag{3.2}$$

In particular, when the functional L is constant on solution $u(t)$, $\|u_t(t)\|^2_{\mathcal{L}^2} = 0$, so $u_t(t,x) = 0$ a.e. in $\mathbb{R}^n$ for $t > 0$. So for $v \in H^1(\mathbb{R}^n)$ if $L(S(t)v) = c \in \mathbb{R}$ for all $t \ge 0$, then $S(t)v = v$ for all $t \ge 0$.

d. Using Cauchy-Schwarz

$$\begin{aligned} L(u) &\ge \|\nabla u\|^2_{\mathbb{L}^2} + c_1(\alpha,\mu)\|u\|^2_{\mathbb{R}^n} - c_2(\alpha,\mu)\|C_\mu\|^2_{\mathcal{L}^2} - 2\|\phi\|_{\mathcal{L}^1} & (3.3)\\ &\ge \min\{1, c_1(\alpha,\mu)\}\|u\|^2_{H^1} - c_2(\alpha,\mu)\|C_\mu\|^2_{\mathcal{L}^2} - 2\|\phi\|_{\mathcal{L}^1} & (3.4)\end{aligned}$$

So if we have coercivity, i.e. if $\|u\|_{H^1} \to \infty$ then $L(u) \to \infty$

So indeed L defines a Lyapunov function for 1.5.

3.2 Stationary Solution Estimates

In order to estimate the stationary solutions we need to assume a condition similar to 1.3 for $g(x,u)$. That is, there is a $\delta \in (0,\alpha)$ and a non-negative $\mathcal{L}^2(\mathbb{R}^n)$ function $C_\delta(x)$, that depends on δ and a positive $\mathcal{L}^1(\mathbb{R}^n)$, $\phi_1(x)$ such that

$$g(x,u) \le \delta u^2 + C_\delta(x)|u| + \phi_1(x). \tag{3.5}$$

A stationary solution v, is a weak solution of the problem $\Delta v + f(x,v) = 0$. Multiplying by v and integrating over $\mathbb{R}^n$ we have the estimate,

$$\|\nabla v\|^2_{\mathcal{L}^2} \le (\alpha-\delta)\|v\|^2_{\mathcal{L}^2} + \|C_\delta\|_{\mathcal{L}^2}\|v\|_{\mathcal{L}^2} + \|\phi_1\|_{\mathcal{L}^1} \tag{3.6}$$

Using the Cauchy-Schwarz,

$$\|\nabla v\|^2_{\mathcal{L}^2} + (\alpha-\delta-\xi)\|v\|^2_{\mathcal{L}^2} \le \frac{1}{4\xi}\|C_\delta\|^2_{\mathcal{L}^2} + \|\phi_1\|_{\mathcal{L}^1} \tag{3.7}$$

where ξ is a constant such that $\alpha - \delta - \xi > 0$. As 3.1 is a Lyapunov function for 1.5 and with the assumptions 3.5 the set of the stationary solutions is bounded in H^1. As a consequence 1.5 generates a gradient system in $H^1(\mathbb{R}^)$. The boundedness of the stationary solutions tell us that the semigroup of the global solutions is point dissipative in $H^1(\mathbb{R}^n)$. Point dissipative implies that there is a maximal compact invariant set which attracts bounded sets. Thus we have a global attractor, call this $\mathcal{A}$.

4 An Estimate for the dimension of the attractor

In the usual Sobolev space setting, the negative Laplacian has absolutely continuous spectrum $\sigma(-\Delta) = [0, \infty)$. It is invertible; however, the inverse is unbounded. This creates problems in finding a bound on the global attractor. For simplicity let $f(x, u) = f(u)$. As this is the only nonlinear term in the equation $\Delta f(u)$. The linearization $f(u)$ about an orbit is $f'(u(t))u$. Let $w = -\Delta u$ then $w_t = -\Delta u_t$ now define

$$z = \begin{pmatrix} u \\ w \end{pmatrix}, \quad A = \begin{pmatrix} 1-\nu & -\nu \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \Delta + f'(u) \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

The linearization for equation 1.5 about an orbit in a compact form is written as

$$Az_t = B(u)z \tag{4.2}$$

As $\det(A) = 1$ we can write the system as

$$z_t = A^{-1}B(u)z = \begin{pmatrix} 0 & \Delta + f'(u) \\ 0 & (1-\nu)\Delta + f'(u) \end{pmatrix} z \tag{4.3}$$

Hence the linearized system also form a semigroup

$$z(t) = \exp(A^{-1}B(u)t)z_0, \tag{4.4}$$

where $z_0 = (u_0, \Delta u_0)$. Now evolution for the n-dimensional volume element is given by

$$\frac{d}{dt} \ln \| \bigwedge x_i \| \le \sum_{i=k}^{n} \Re \langle u_k, A^{-1}B(u)U_k \rangle \tag{4.5}$$

Now the spectrum of $\sigma(\Delta) = (-\infty, 0]$ this is a continuous spectrum. Recalling the bounds for f given by 1.3, 3.5

$$\frac{d}{dt} \ln \| \bigwedge x_i \| \le -\int_0^n (1-\nu)x^2 + \alpha + \delta \|u\|_{\mathcal{L}^2} + \|C_\delta u + \phi\|_{\mathcal{L}^1} \, dx \tag{4.6}$$

And so if

$$n > \frac{3}{1-\nu}(\alpha + \delta\|u\|_{\mathcal{L}^2} + \|C_\delta u + \phi\|_{\mathcal{L}^1}) \tag{4.7}$$

We have a an upper bound on the dimension. As this can be quite large and depends on the initial conditions, the bound does not seem to useful. The dimension really depends on the non-linearity f.

The main theorem of [4] is:

Theorem 1 *Under the given assumptions, For any $M > \frac{1}{4\xi}\|C_\delta\|^2_{\mathcal{L}^2} + \|\phi_1\|_{\mathcal{L}^1}$, set*

$$B_0 = \cup_{T \ge 0} S(t)B_1, \quad B_1 = \{u_0 \in H^1(\mathbb{R}^n) : \|\nabla u_0\|^2_{\mathcal{L}^2} + (\alpha - \delta - \eta)\|u_0\|^2_{\mathcal{L}^2} \le M, \|(-\Delta)^{1/2}u_0\|_{\mathcal{L}^2} < \infty\} \tag{4.8}$$

Then there exists positive constants M_0 and T such that

$$\|S(t)B_0\|_{H^2} \le M < \infty, \quad \forall t \ge T. \tag{4.9}$$

Despite having similar results, the approach they use is quite different. The authors in [4] consider a restricted phase space using energy estimates with the semigroup $S(t)$ and the Dirac operator $(-\Delta)^{1/2}$.

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